*(JZS) Journal of Zankoy Sulaimani, December 2008, 10(1) Part A(29-39)*
گۆڤاری زانکۆی سلێمانی ،کانونی یەکەمی 2008 بەرگ 10(1) بەشی A لاپەرە (29-39)

# Some Effects of $(\theta,\varphi)-$Derivations on Centrally Prime Rings.

* **Adil Kadir Jabbar,** Mathematics Department, College of Science, University of Sulaimani, Kurdistan Region\ Iraq

**Abdulrahman Hamed Majeed**, Mathematics Department, College of Science, University of Baghdad, Iraq.

## Abstract

*In this paper we study some effects of $(\theta,\varphi)-$derivations on centrally prime rings, and we try to extend some results on prime rings which are concerned with $(\theta,\varphi)-$derivations to centrally prime rings and also we determine those conditions under which these extensions are allowed.*

**Keywords**: Derivations, prime rings, centrally prime rings, Lie ideals.

## The Fundamentals:

Let $R$ be a ring .A non-empty subset $S$ of $R$ is said to be a multiplicative system in $R$ if $0\notin S$ and $a,b\in S$ implies $ab\in S$, [1]. Let $S$ be a multiplicative system in $R$ such that $[S,R]=\{0\}$,where $[S,R]=\{[s,r]:s\in S,r\in R\}$ and $[s,r]$ is the commutator defined by $sr-rs$. Define a relation (~) on $R\times S$ as follows :

If $(a,s)$ and $(b,t)$ are in $R\times S$, then $(a,s)\sim(b,t)$ if and only if there exists $x\in S$ such that $x(at-bs)=0$ .Since $[S,R]=\{0\}$. We will show that (~) is an equivalence relation on $R\times S$.

**Reflexivity:**

if $(a,s)\in R\times S$, where $a\in R$ and $s\in S$, then $s(as-as)=0$, so $(a,s)\sim(a,s)$, and hence (~) is a reflexive relation on $R\times S$.

**Symmetry:**

Let for $(a,s),(b,t)\in R\times S$ we have $(a,s)\sim(b,t)$, where $a,b\in R$ and $s,t\in S$, then there exists $k\in S$ such that $k(at-bs)=0$, which gives $k(bs-at)=0$, and hence $(b,t)\sim(a,s)$and thus (~) is a symmetric relation on $R\times S$.

**Transitivity:**

If $(a,s),(b,t),(r,u)\in R\times S$, are such that $(a,s)\sim(b,t)$ and $(b,t)\sim(r,u)$, where $a,b,r\in R$ and $s,t,u\in S$ then there exists $k,l\in S$, such that $k(at-bs)=0$ and $l(bu-rt)=0$, hence we get $luk(at-bs)=0$ and $ksl(bu-rt)=0$ that is,

$$lukat=lukbs\text{......}\mathbf{(1)}$$

and

$$kslbu=kslrt\text{.......}\mathbf{(2).}$$

But since $[S,R]=\{0\}$, so $lukbs=kslbu$ and $kslrt=lktrs$, and now use these in **(1)** and **(2)** we get $lktau=lktrs$, or $lkt(au-rs)=0$, where $lkt\in S$ which means that $(a,s)\sim(r,u)$, thus (~) is a transitive relation on $R\times S$. Hence (~) is an equivalence relation on $R\times S$.

Now denote the equivalence class of $(a,s)$ in $R\times S$ by $a_S$, and the set of all equivalence classes determined under this

---

***E-mail: adil_math_sci@yahoo.com**

equivalence relation by $R_S$, that is, $a_S = \{(b,t) \in R \times S : (a,s) \sim (b,t)\}$ and $R_S = \{a_S : (a,s) \in R \times S\}$. We define addition (+) and multiplication (.) on $R_S$ as follows: $a_S + b_t = (at + bs)_{st}$ and $a_S . b_t = (ab)_{st}$, for all $a_S, b_t \in R_S$. It can be shown that these two operations are well-defined and that $(R_S, +, .)$ forms a ring which is called the localization of $R$ at $S$.

**Some Remarks**

We mention to the following remarks which can be found in [1,2,3].

**(i):** $R_S$ has the identity element though $R$ does not have, in fact if $s \in S$ then $s_S$ is the
identity element of $R_S$, and $s_S = t_t$, for all $s, t \in S$.

**(ii):** If $a, b \in R$ and $s \in S$, then $a_S + b_S = (a+b)_S$, and if $a_1, a_2, \ldots, a_n \in R$, then $(a_1)_S + (a_2)_S + \ldots + (a_n)_S = (a_1 + a_2 + \ldots + a_n)_S$.

**(iii):** $0_S$ is the zero of $R_S$ for all $s \in S$, and $0_S = 0_t$, for all $s, t \in S$.

**(iv):** If $a_S \in R_S$, where $a \in R$ and $s \in S$ then $(-a)_S$ is the additive inverse of $a_S$.

**(v):** If $a_S = 0$ in $R_S$, where $a \in R, s \in S$, then there exists $t \in S$ such that $ta = 0$.

**(vi):** If $R$ is a commutative ring with identity then the equivalence class $a_S$ is also denoted by $\frac{a}{s}$ [1] or by $s^{-1}a$ [3], and $R_S$ is also denoted by $S^{-1}R$ [1,3].

**Some Basic Definitions**

Let $R$ be a ring.Then:

**(i):** $R$ is called a prime ring if, whenever $a, b \in R$ such that $aRb = \{0\}$ then $a = 0$ or $b = 0$, and it is called a semiprime ring if $aRa = \{0\}$, then we have $a = 0$, where $aRb = \{arb : r \in R\}$ [4]. It can be concluded that a prime ring is semiprime.

**(ii):** An additive mapping $D : R \to R$ is called a derivation on $R$ if $D(ab) = D(a)b + aD(b)$, for all $a, b \in R$ [5,6].

**(iii):** If $n$ is a positive integer then $R$ is called an $n-$torsion free ring if for $x \in R$, $nx = 0$ implies $x = 0$ [7].

**(iv):** If $A \subseteq R$, then by $A_S$ we mean the set $A_S = \{a_S : a \in A, s \in S\}$.

**(v):** An additive subgroup $U$ of $R$ is said to be a Lie ideal of $R$ if $[U, R] \subseteq U$ [8,9].

**(vi):** $R$ is called a centrally prime ring if $R_S$ is a prime ring for each multiplicative system $S$ in $R$ with $[S, R] = \{0\}$ [2].

**(vii):** A derivation $D : R \to R$ is called a centrally-zero derivation on $R$ if $D(S) = \{0\}$ for each multiplicative system $S$ in $R$ with $[S, R] = \{0\}$ [2].

**(viii):** $R$ is said to satisfy the central commutativity property $(CCP)$ if $R_S$ is a commutative ring for each multiplicative system $S$ in $R$ with $[S, R] = \{0\}$ [2].

**(x):** If $\theta, \varphi$ are two endomorphisms of $R$, then an additive mapping $D : R \to R$ is called a $(\theta, \varphi)-$ derivation on $R$ if

$D(xy) = D(x)\varphi(y) + \theta(x)D(y),$ for all $x, y \in R$, and if $A$ is a nonempty subset of $R$ then an additive mapping $F : R \to R$ is called a generalized $(\theta, \varphi)-$derivation on $A$ if there exists a $(\theta, \varphi)-$derivation $D : R \to R$ such that $F(xy) = F(x)\theta(y) + \varphi(x)D(y),$ for all $x, y \in A$ [10,11].

**(xi):**If $D : R \to R$ is a mapping then by $D^2$ we mean $D \circ D$.In general, if $n$ is any positive integer, then $D^n$ will mean $D \circ D \circ ... \circ D$ ($n$ times). Finally if $x \in R$ then by $xD$ we mean the mapping $xD : R \to R$ which is defined by $(xD)(r) = x(D(r)),$ for all $r \in R$.

Now we mention to the following results,
The proofs of which can be found in the Indicated references.

**Lemma A** [2]

Let $R$ be a ring and $S$ a multiplicative system in $R$ such that $[S, R] = \{0\}$. If $D : R \to R$ is a centrally zero derivation on $R$, then $D_* : R_S \to R_S$, defined by $D_*(r_S) = (D(r))_S$, for all $r_S \in R_S$, is a derivation on $R_S$.(is called the induced derivation by $D$).

**Theorem B** [12]

Let $R$ be a 2-torsion free semiprime ring , $U$ a Lie ideal of $R$ and $D$ a derivation of $R$ such that $D^n(u) = 0$, for all $u \in U$, where $n \geq 1$, is a fixed integer, then $D(U) = \{0\}$.

Since every prime ring is semiprime so **Theorem B,** remains valid if we replace the word " semiprime " by the word " prime ".

**Theorem C** [11]

Let $R$ be a ring 2-torsion free prime ring , $U$ a Lie ideal of $R$ such that $u^2 \in U$, for all $u \in U$.Suppose that $\theta$ is an automorphism of $R$.If $D : R \to R$ is an additive mapping satisfying $D(u^2) = 2\theta(u)D(u)$, for all $u \in U$, then either $D(U) = \{0\}$ or $U \subseteq Z(R)$.

**Theorem D** [10]

Let $R$ be a 2-torsion free prime ring , $U$ a non zero Lie ideal of $R$ such that $u^2 \in U$, for all $u \in U$. Suppose that $\theta$ is an automorphism of $R$ and $F : R \to R$ is a generalized $(\theta, \theta)-$derivation associated with a derivation $D$. Then:

**(i):** If $F$ acts as a homomorphism on $U$, then either $D = 0$ on $R$ or $U \subseteq Z(R)$.

**(ii):** If $F$ acts as anti-homomorphism on $U$, then either $D = 0$ on $R$ or $U \subseteq Z(R)$.

**Theorem E** [4]

Let $R$ be a prime ring and $K$ a non zero ideal of $R$ and let $\theta, \varphi$ be automorphisms of $R$. Suppose that $D : R \to R$ is a $(\theta, \varphi)-$ derivation of $R$.

**(i):** If $D$ acts as a homomorphism on $K$, then $D = 0$ on $R$.

**(ii):** If $D$ acts as anti-homomorphism on $K$, then $D = 0$ on $R$.

Throughout this paper all rings under consideration are with non-zero center, $Z(R)$.

## The Main Results

First we prove two lemmas which will lead to the first result of this paper.

**Lemma 1**

Let $R$ be a ring in which $Z(R)$ contains no proper zero divisors of $R$ then $Z(R) - \{0\}$ is a multiplicative system in $R$ and $[Z(R) - \{0\}, R] = \{0\}$.

**Proof**

It is clear that $0 \notin Z(R)-\{0\}$.Now let $a,b \in Z(R)-\{0\}$, then $ab \in Z(R)$ and since $Z(R)$ has no proper zero divisors of $R$ and $a \neq 0, b \neq 0$ so $ab \neq 0$, that is $ab \in Z(R)-\{0\}$. Hence $Z(R)-\{0\}$ is a multiplicative system in $R$ and since $Z(R)$ is a commutative subring of $R$, one can easily show that $[Z(R)-\{0\}, R] = \{0\}$.

**Lemma 2**

If $R$ is a ring which satisfies $(CCP)$ and $Z(R)$ contains no proper zero divisors then $R$ is commutative.

**Proof**

By **Lemma 1,** we have $S = Z(R)-\{0\}$ is a multiplicative system in $R$ and $[S,R]=\{0\}$ and since $R$ satisfies $(CCP)$ so $R_S$ is commutative. To show $R$ is commutative. Let $x,y \in R$, then fix $s \in S$, and so $x_s, y_s \in R_S$. Hence $x_s y_s = y_s x_s$ and then we get $(xy-yx)_{ss} = (xy)_{ss} - (yx)_{ss} = x_s y_s - y_s x_s = 0$, which means there exists $t \in S$ such that $t(xy-yx)=0$. Since $Z(R)$ contains no proper zero divisor of $R$ so $xy-yx=0$, that is $xy=yx$ and thus $R$ is commutative.

Let us introduce the following definition.

**Definition**

Let $R$ be a ring. We call a Lie ideal $U$ of $R$ a centrally closed Lie ideal of $R$ if $US \subseteq U$ for each multiplicative system $S$ of $R$ with $[S,R]=\{0\}$, where $US = \{us : u \in U, s \in S\}$.

Now we prove the following result.

**Lemma 3**

Let $R$ be a ring and $S$ a multiplicative system in $R$ with $[S,R]=\{0\}$.If $U$ is a centrally closed Lie ideal of $R$, then $U_S$ is a Lie ideal of $R_S$.

**Proof**

$S \neq \phi$ implies that there exists $s \in S$. Since $0 \in U$ so $0_s \in U_S$ and thus $\phi \neq U_S \subseteq R_S$. If $u_s, v_t \in U_S$, where $u,v \in U$ and $s,t \in S$. Then we have $u_s - v_t = u_s + (-v)_t = (ut + (-vs))_{st} = (ut - vs)_{st} \in U_S$ (Since $ut, vs \in U$ so $ut - vs \in U$ and $s,t \in S$ implies $st \in S$). Hence $U_S$ is a subgroup of $R_S$.To show $[U_S, R_S] \subseteq U_S$.Let $u_s \in U_S, r_t \in R_S$, where $u \in U, r \in R, s,t \in S$.
Then
$[u_s, r_t] = u_s r_t - r_t u_s = (ur - ru)_{st} = ([u,r])_{st} \in U_S$ (Since $[U,R] \subseteq U$ and $st \in S$). Hence $[U_S, R_S] \subseteq U_S$, and thus $U_S$ is a Lie ideal of $R_S$.

Now we are able to give the first result of the paper.

**Theorem 4**

Let $R$ be a 2-torsion free centrally prime ring in which $Z(R)$ has no proper zero divisors and $U$ is a centrally closed Lie ideal of $R$.If $R$ admits a centrally zero derivation $D: R \to R$ such that $D^n(u)=0$, for all $u \in U$, where $n \geq 1$ is a fixed integer, then $D(u)=0$, for all $u \in U$.

**Proof**

By **Lemma 1,** we have $S = Z(R)-\{0\}$ is a multiplicative system in $R$ with $[S,R]=\{0\}$. Now we have $R_S$ is a 2-

torsion free prime ring, and by **Lemma 3,** $U_S$ is a Lie ideal of $R_S$. Since $D$ is a centrally zero derivation on $R$ so by **Lemma A,** $D_*: R_S \to R_S$, which is defined by $D_*(r_s) = (D(r))_s$, is a derivation on $R_S$**.** Now if $u_s \in U_S$ is any element, where $u \in U, s \in S$, then we have $D_*^n(u_s) = (D^n(u))_s = 0_s = 0$. Hence $R_S$ is a 2-torsion free prime ring (and hence a 2- torsion free semiprime ring), $U_S$ is a Lie ideal of $R_S$ such that $D_*$ is a derivation of $R_S$ with $D_*^n(u_s) = 0$, for all $u_s \in U_S$. Thus by **Theorem B,** we get $D_*(u_s) = 0$, for all $u_s \in U_S$. Now fix an element $t \in S$. If $x \in U$ is any element then $x_t \in U_S$, and hence $(D(x))_t = D_*(x_t) = 0$ which means that there exists $u \in S$ such that $uD(x) = 0$ and since $Z(R)$ contains no proper zero divisors of $R$ and $0 \neq u \in S \subseteq Z(R)$ so we get $D(x) = 0$, and this result is true for all $x \in U$. Thus $D(u) = 0,$ for all $u \in U$.

Now we prove some results which will lead to the second result of this paper.

**Lemma 5**

If $R$ is an $n-$torsion free ring and $S$ is a multiplicative system in $R$ with $[S, R] = \{0\}$, then $R_S$ is also an $n-$torsion free ring.

**Proof**

The proof is trivial.

**Proposition 6**

Let $R$ be a ring and $S$ a multiplicative system in $R$ with $[S, R] = \{0\}$. If $\theta : R \to R$ is an automorphism of $R$ which acts as the identity map on $S$, then $\theta' : R_S \to R_S$, defined by $\theta'(r_s) = (\theta(r))_s$, for all $r_s \in R_S$, is an automorphism of $R_S$. (We call $\theta'$, the induced endomorphism by $\theta$).

**Proof**

Let $a_s, b_t \in R_S$, where $a, b \in R$ and $s, t \in S$.

$\theta'(a_s + b_t) = \theta'((at + bs)_{st}) =$
$(\theta(at + bs))_{st} = (\theta(at) + \theta(bs))_{st} =$
$(\theta(a)t + \theta(b)s)_{st} = (\theta(a)t)_{st} +$
$(\theta(b)s)_{ts} = (\theta(a))_s\, t_t + (\theta(b))_t\, s_s =$
$(\theta(a))_s + (\theta(b))_t = \theta'(a_s) + \theta'(b_t)$.

Also we have $\theta'(a_s b_t) = \theta'((ab)_{st}) =$
$(\theta(ab))_{st} = (\theta(a)\theta(b))_{st} =$
$(\theta(a))_s (\theta(b))_t = \theta'(a_s)\theta'(b_t)$. Hence $\theta'$ is an endomorphism of $R_S$.

To show that $\theta'$ is a bijective mapping. Let $a_s \in Ker\theta'$. Then $\theta'(a_s) = 0$.So that $(\theta(a))_s = 0$. Hence there exists $t \in S$ such that $t\theta(a) = 0$. Then $\theta(t)\theta(a) = 0$. Therefore $\theta(ta) = 0$, so that $ta \in Ker\theta$, and since $\theta$ is one to one, so we get $ta = 0$.Then,

$a_s = t_t a_s = (ta)_{ts} = 0_{ts} = 0$.

So $Ker\theta' = \{0\}$, and hence $\theta'$ is one to one. If $y_u \in R_S$, for $y \in R$ and $u \in S$. Then there exists $x \in R$ such that

$\theta(x) = y$ .Then $x_u \in R_S$ and $\theta'(x_u) = (\theta(x))_u = y_u$ .So that $\theta'$ is onto.Hence $\theta'$ is an automorphism of $R_S$ .

Now we introduce the following definition.

**Definition**

Let $R$ be a ring. We say, a mapping $f : R \to R$ satisfies central identity property $(CIP)$ if for each multiplicative system $S$ in $R$ with $[S, R] = \{0\}$ we have $f(s) = s$, for all $s \in S$ .

Now we give the second result of this paper.

**Theorem 7**

Let $R$ be a 2-torsion free centrally prime ring in which $Z(R)$ has no proper zero divisors and $U$ is a centrally closed Lie ideal of $R$ such that $u^2 \in U$, for all $u \in U$ . If $\theta$ is an automorphism of $R$ which satisfies $(CIP)$ and $D : R \to R$ is an endomorphism which satisfies $(CIP)$ and $D(u^2) = 2\theta(u)D(u)$, for all $u \in U$, then either $D(U) = \{0\}$ or $U \subseteq Z(R)$ .

**Proof**

By **Lemma 1,** we have $S = Z(R) - \{0\}$ is a multiplicative system in $R$ with $[S, R] = \{0\}$ . Also by **Lemma 5,** we have $R_S$ is a 2-torsion free prime ring, and by **Lemma 3,** $U_S$ is a Lie ideal of $R_S$ . Also for all $u_s \in U_S$ we have $(u_s)^2 = (u^2)_{s^2} \in U_S$ (since $u^2 \in U$, for all $u \in U$ and $s^2 \in S$ ).Let $\theta' : R_S \to R_S$, be the induced automorphism of **Proposition 6,** which is defined by $\theta'(r_s) = (\theta(r))_s$, and also define $D' : R_S \to R_S$, by $D'(r_s) = (D(r))_s$ for all $r_s \in R_S$ . Now to show that $D'$ is an additive mapping. Let $a_s, b_t \in R_S$, where $a, b \in R$ and $s, t \in S$ . We have

$D'(a_s + b_t) = D'((at + bs)_{st}) =$
$(D(at + bs))_{st} = (D(at) + D(bs))_{st} =$
$(D(a)D(t) + D(b)D(s))_{st} = (D(a)t +$
$D(b)s)_{st} = (D(a)t)_{st} + (D(b)s)_{ts} =$
$(D(a))_s t_t + (D(b))_t s_s = (D(a))_s +$
$(D(b))_t = D'(a_s) + D'(b_t)$ . So $D'$ is an additive mapping. If $u_s \in U_S$ is any element, where $u \in U$ and $s \in S$, then we have

$D'((u_s)^2) = D'((u^2)_{s^2}) = (D(u^2))_{s^2} =$
$(2\theta(u)D(u))_{ss} =$
$(\theta(u)D(u) + \theta(u)D(u))_{ss} =$
$(\theta(u)D(u))_{ss} + (\theta(u)D(u))_{ss} =$
$(\theta(u))_s(D(u))_s + (\theta(u))_s(D(u))_s =$
$\theta'(u_s)D'(u_s) + \theta'(u_s)D'(u_s) =$
$2\theta'(u_s)D'(u_s)$ . Thus $R_S$ is a 2-torsion free prime ring, $U_S$ is a Lie ideal of $R_S$ with $(u_s)^2 \in U_S$, for all $u_s \in U_S$ .Also $\theta'$ is an automorphism of $R_S$ and $D'$ is an additive mapping of $R_S$ such that

$D'((u_s)^2) = 2\theta'(u_s)D'(u_s)$, for all $u_s \in U_S$ .Hence by **Theorem C,** we get either $D'(U_S) = \{0\}$ or $U_S \subseteq Z(R_S)$. Let $D'(U_S) = \{0\}$. To show $U = 0$. Let $u \in U$ be any element, then for a fixed $s \in S$ we have $(D(u))_s = D'(u_s) = 0$ so there exists $t \in S$ such that $tD(u) = 0$ and since $Z(R)$ contains no proper zero divisors of $R$ so $D(u) = 0$ which means that $D(U) = \{0\}$. If $U_S \subseteq Z(R_S)$. To show that $U \subseteq Z(R)$. Let $v \in U$, and fix $t \in S$, then for all $r \in R$, we have $[v_t, r_s] = 0$ and hence

$([v,r])_{ts} = [v_t, r_s] = 0$, which implies that there exists $w \in S$ such that $w[v,r] = 0$, and since $Z(R)$ contains no proper zero divisors of $R$ so we get $[v,r] = 0$. Hence $U \subseteq Z(R)$.

Now we prove some results which will lead to the next result of the paper.

**Proposition 8**

Let $R$ be a ring and $S$ a multiplicative system in $R$ with $[S,R] = \{0\}$. If $\theta, \varphi$ are two endomorphisms of $R$ which act as the identity map on $S$ and $D$ is a $(\theta,\varphi)-$derivation such that $D(S) = \{0\}$, then $D' : R_S \to R_S$, defined by $D'(r_s) = (D(r))_s$, for all $r_s \in R_S$ is a $(\theta',\varphi')-$derivation of $R_S$, where $\theta', \varphi'$ are endomorphisms of $R_S$.

**Proof**

It is easy to show that $D'$ is an additive mapping. Now consider the induced endomorphisms $\theta'$ and $\varphi'$ of **Proposition 6,** which formed from the endomorphisms $\theta$ and $\varphi$. Then we have

$D'(a_s b_t) = D'((ab)_{st}) = (D(ab))_{st} =$
$(D(a)\varphi(b) + \theta(a)D(b))_{st} =$
$(D(a)\varphi(b))_{st} + (\theta(a)D(b))_{st} =$
$(D(a))_s(\varphi(b))_t + (\theta(a))_s(D(b))_t =$
$D'(a_s)\varphi'(b_t) + \theta'(a_s)D'(b_t)$. Hence $D'$ is a $(\theta',\varphi')-$derivation of $R_S$. (Note that we call $D'$ the induced $(\theta',\varphi')-$derivation by the $(\theta,\varphi)-$ derivation $D$).

**Corollary 9**

Let $R$ be a ring and $S$ a multiplicative system in $R$ with $[S,R] = \{0\}$. If $\theta, \varphi$ are two automorphisms of $R$ which act as the identity map on $S$ and $D$ is a $(\theta,\varphi)-$derivation such that $D(S) = \{0\}$, then $D' : R_S \to R_S$, defined by $D'(r_s) = (D(r))_s$, for all $r_s \in R_S$ is a $(\theta',\varphi')-$derivation of $R_S$.

**Proof**

Since every automorphism of a ring is also an endomorphism, so by applying **Proposition 8,** the result will follows.

**Proposition 10**

Let $R$ be a ring and $S$ a multiplicative system in $R$ with $[S,R] = \{0\}$. If $\theta$ is an automorphism of $R$ which acts as the identity map on $S$ and $F : R \to R$ is a generalized $(\theta,\theta)-$derivation with the associated $(\theta,\theta)-$derivation $D$, such that $D(S) = \{0\}$, then $F' : R_S \to R_S$, defined by $F'(r_s) = (F(r))_s$, for all $r_s \in R_S$, is a generalized $(\theta',\theta')-$ derivation associated with the $(\theta',\theta')-$derivation $D'$.

**Proof**

Since $D$ is a $(\theta,\theta)-$derivation and $F$ a generalized $(\theta,\theta)-$derivation associated with $D$, so that $D:R\to R$ and $F:R\to R$ are additive mappings such that $D(xy)=D(x)\theta(y)+\theta(x)D(y),$ for all $x,y\in R$, and $F(xy)=F(x)\theta(y)+\theta(x)D(y),$ for all $x,y\in R$.Since $\theta$ is an automorphism of $R$ which acts as the identity map on $S$, so by **Proposition 6,** $\theta':R_S\to R_S$ which is defined by $\theta'(r_s)=(\theta(r))_s$, is an automorphism of $R_S$.Then by **Corollary 9,** $D':R_S\to R_S$, which is defined by $D'(r_s)=(D(r))_s$, is a $(\theta',\theta')-$derivation. To show $F'$ is a generalized $(\theta',\theta')-$derivation with the associated $(\theta',\theta')-$derivation $D'$.Let $a_s,b_t\in R_S$ for $a,b\in R$ and $s,t\in S$. Now if $a_s=b_t$ then there exists $k\in S$ such that $k(at-bs)=0$, hence $kat=kbs$, and now $F'(a_s)=F'(k_k a_s t_t)=F'((kat)_{kst})=F'((kbs)_{kts})=F'(k_k b_t s_s)=F'(b_t)$. So $F'$ is well defined. As in **Theorem 7,** we can show that $F'$ is an additive mapping. Finally, we have $F'(a_s b_t)=F'((ab)_{st})=(F(ab))_{st}=(F(a)\theta(b)+\theta(a)D(b))_{st}=(F(a)\theta(b))_{st}+(\theta(a)D(b))_{st}=(F(a))_s(\theta(b))_t+(\theta(a))_s(D(b))_t=F'(a_s)\theta'(b_t)+\theta'(a_s)D'(b_t)$.Hence $F'$ is a generalized $(\theta',\theta')-$derivation with the associated $(\theta',\theta')-$derivation $D'$.

Now it is the time to introduce the following definition.

**Definition**

By a centrally zero $(\theta,\varphi)-$derivation we mean a $(\theta,\varphi)-$derivation which also is centrally zero.

**Theorem 11**

Let $R$ be a 2-torsion free centrally prime ring in which $Z(R)$ contains no proper zero divisors of $R$ and $U$ be a nonzero centrally closed Lie ideal of $R$ with $u^2\in U$, for all $u\in U$. Suppose $\theta$ is an automorphism of $R$ and $F:R\to R$ is a generalized $(\theta,\theta)-$derivation with the associated centrally zero $(\theta,\theta)-$derivation $D$. Then:

**(i):** If $F$ acts as a homomorphism on $U$, then either $D=0$ on $R$ or $U\subseteq Z(R)$.

**(ii):** If $F$ acts as anti-homomorphism on $U$, then either $D=0$ on $R$ or $U\subseteq Z(R)$.

**Proof**

Since $D$ is a $(\theta,\theta)-$derivation and $F$ is a generalized $(\theta,\theta)-$derivation associated with $D$, so $D:R\to R$ and $F:R\to R$ are additive mappings such that $D(xy)=D(x)\theta(y)+\theta(x)D(y),$ for all $x,y\in R$, and $F(xy)=F(x)\theta(y)+\theta(x)D(y),$ for all $x,y\in R$. Now by **Lemma 1,** we have $S=Z(R)-\{0\}$ is a multiplicative system in $R$ with $[S,R]=\{0\}$. Since $R$ is a 2-torsion free centrally prime ring so by **Lemma 5,** we get $R_S$ is a 2-torsion free prime ring and by **Lemma 3,** we get $U_S$ is a Lie ideal of $R$. Since $S\neq\phi$, fix an $s\in S$. If $U_S=0$, then for any $u\in U$, we have $u_s\in U_S$, and thus $u_s=0$, which means there exists $t \in S$ such that $tu = 0$, and as $0 \neq t \in S \subseteq Z(R)$ and $Z(R)$ contains no proper zero divisors of $R$ we get $u = 0$, and this result is true for all $u \in U$, so that $U = 0$ which is a contradiction and hence $U_S \neq 0$. Now if $u_s \in U_S$, where $u \in U, s \in S$, then $(u_s)^2 = u^2_{s^2} \in U_S$ (Since $u^2 \in U, s^2 \in S$). Also by **Proposition 10,** we have $F' : R_S \to R_S$, defined by $F'(r_s) = (F(r))_s$, is a generalized $(\theta', \theta')-$derivation with the associated $(\theta', \theta')-$derivation $D'$.Thus $R_S$ is a 2-torsion free prime ring and $U_S$ is a nonzero Lie ideal of $R_S$ with $(u_s)^2 \in U_S$, for all $u_s \in U_S$. That is, $\theta'$ is an automorphism of $R_S$ and $F'$ is a generalized $(\theta', \theta')-$ derivation with the associated $(\theta', \theta')-$ derivation $D'$.

**(i):** If $F$ acts as a homomorphism on $U$. To show $F'$ acts as a homomorphism on $U_S$. So let $u_s, v_t \in U_S$, where $u, v \in U$ and $s, t \in S$.Then since $F'$ is additive so $F'(u_s + v_t) = F'(u_s) + F'(v_t)$, and also we have $F'(u_s v_t) = F'((uv)_{st}) = (F(uv))_{st} = (F(u)F(v))_{st} = (F(u))_s (F(v))_t = F'(u_s)F'(v_t)$. Hence $F'$ acts as a homomorphism on $U_S$. Hence by **Theorem D,** we get either $D' = 0$ on $R_S$ or $U_S \subseteq Z(R_S)$. As $S \neq \phi$, fix an $s \in S$. If $D'(R_S) = \{0\}$, to show $D(R) = \{0\}$.Let $r \in R$ be any element, then $r_s \in R_S$. So $(D(r))_s = D'(r_s) = 0$.Since $Z(R)$ contains no proper zero divisors it can be shown that $D(r) = 0$, which in consequence, implies that $D(R) = \{0\}$, that is $D = 0$.If $U_S \subseteq Z(R_S)$, to show $U \subseteq Z(R)$.Let $u \in U$ and let $a \in R$ be any element. Then $u_s \in U_S \subseteq Z(R_S)$, and $a_s \in R_S$.Hence $([u, a])_{ss} = [u_s, a_s] = 0$.So there exists $t \in S$ such that $t[u, a] = 0$, where $0 \neq t \in S \subseteq Z(R)$, and as $Z(R)$ contains no proper zero divisors we get $[u, a] = 0$, which means, $[U, R] = \{0\}$, that is, $U \subseteq Z(R)$.

**(ii):** If $F$ acts as anti-homomorphism on $U$, then as the same as in **(i),** we can prove that $F'$ acts as anti-homomorphism on $U_S$, and hence either $D' = 0$ or $U_S \subseteq Z(R_S)$. Again as in **(i),** we get either $D = 0$ or $U \subseteq Z(R)$.

**Theorem 12**

Let $R$ be a centrally prime ring in which $Z(R)$ contains no proper zero divisors of $R$ and $K$ is a nonzero ideal of $R$.Suppose that $\theta, \varphi$ are automorphisms of $R$ satisfy $(CIP)$ and $D : R \to R$ is a centrally zero $(\theta, \varphi)-$ derivation of $R$.

**(i):** If $D$ acts as a homomorphism on $K$, then $D = 0$ on $R$.

**(ii):** If $D$ acts as anti-homomorphism on $K$, then $D = 0$ on $R$.

**Proof**

By **Lemma 1,** $S = Z(R) - \{0\}$ is a multiplicative system in $R$ with $[S, R] = \{0\}$. Fix an $s \in S$ (Since $S \neq \phi$). Then by by **Proposition 6,** $\theta', \varphi'$ are automorphisms of $R_S$ and by **Corollary 9,** $D'$ is a $(\theta', \varphi')$ – derivation.If $K_S = 0$, then for any $k \in K$, we have $k_S = 0$, and hence there exists $t \in S$ such that $tk = 0$.Since $0 \neq t \in S \subseteq Z(R)$ and $Z(R)$ has no proper zero divisors, so we get $k = 0$, that means $K = 0$ which is a contradiction. Thus $R_S$ is a prime ring, $K_S$ is a nonzero ideal of $R_S$, and $\theta', \varphi'$ are automorphisms of $R_S$ with $D'$ as a $(\theta', \varphi')-$ derivation of $R_S$.

**(i):** If $D$ acts as a homomorphism on $K$, to show $D'$ acts as a homomorphism on $K_S$.Now for all $a_s, b_t \in K_S$, where $a, b \in K$ and $s, t \in S$ we have $D'(a_s b_t) = D'((ab)_{st}) = (D(ab))_{st} = (D(a)D(b))_{st} = (D(a))_s (D(b))_t = D'(a_s)D'(b_t)$. Hence by **Theorem E,** we get $D'(R_S) = \{0\}$.Now if $r \in R$ is any element, then we have $r_S \in R_S$.Thus $(D(r))_S = D'(r_S) = 0$. Hence there exists $t \in S$ such that $tD(r) = 0$, and as $Z(R)$ has no proper zero divisors we get $D(r) = 0$, and hence $D(R) = \{0\}$, that is $D = 0$ on $R$.

**(ii):** If $D$ acts as anti-homomorphism on $K$, by using the same technique as in **(i),** we can show $D'$ acts as anti-homomorphism on $K_S$, and hence we get, as before, that $D = 0$ on $R$.

## References

1. Larsen,M.D. and McCarthy,P.J.: Multiplicative Theory of Ideals, Academic Press New York and London, **1971.**
2. Jabbar,A. K.: Centrally prime rings which are commutative, *Journal of Kirkuk University*, **2006,** 1(2)**,** 108-124.
3. .Ranicki,A.: Noncommutative Localization in Algebra and Topology, London Mathematical Society , Cambridge university press **2006.**
4. Ashraf,M.,Rehman,N.and Quadri,M.A.:On $(\sigma, \tau)-$ derivations in certain rings, Rad. Mat. **1999,** 9, 187-192.
5. Jung, Y.S. and Park, K.H.,: On Generalized $(\alpha, \beta)-$Derivations and commutativity in Prime Rings, *Bull.Korean Math.Soc*. **2006,** 43 (1)**,** pp 101-106.
6. Vukman, J.: An identity related to centralizers in semiprime rings, *Comment Math.Univ. Carolinae***, 1999,** 40 (3), 447-456.
7. Vukman, J.: Centralizers on semiprime rings, *Comment.Math. Univ. Carolinae*, **2001,** 42 (2) ,237-245.

8. Haetinger, C.: Higher Derivations on Lie Ideals, *Tendencias em Matematica Aplicada Computacional*, **2002,** 3 (1)**,** 141-145.
9. Kaya, K., Golbasi, O and Aydin, N.: Some Results for Generalized Lie Ideals in Prime Rings with Derivation II$^*$, *Applied Mathematics*, **2001, 1,** 24-30.
10. Ali, A.,and Kumar, D.**:** Derivation which acts as a homomorphism or as anti-homomorphism in a prime ring , *International Mathematical Forum* **, 2007,** 2(23)**,** 1105-1110.
11. Ashraf, M. **:** On Left $(\vartheta,\phi)-$Derivations of Prime Rings, *Archivum Mathematicum (BRNO) Tomus,* **2005,** 41, pp 157-166.
12. Carini, L., and Giambruno, A.: Lie ideals and nil derivations, *Boll. Un. Mat. Ital*. **1985,** 6, 497-503.

## هەندێ کاریگەری داتاشراوەکانی $(\theta، \varphi)$ لەسەرجۆغزە بەناوەند سەرەتاییکان

**عادل قادرجبار, بەشی ماتماتیك-کۆلێجی زانست-زانکۆی سلێمانی-هەرێمی کوردستان\ عێراق**
**عبدالرحمن حميد مجيد, بەشی ماتماتیك-کۆلێجی زانست- زانکۆی بەغداد- عێراق**

### پوختە

هەوڵی گشتاندنی هەندێ و لەم توێژینەوەیەلێکۆڵینەوەی هەندێك کاریگەرەکانی داتاشراوەکانی $(\varphi، \theta)$ لەسەرجۆغزەبەناوەند سەرەتاییکان ئەنجام درا ناوەند سەرەتاییکان دەدەین وئەومەرجانەدیاری بە جۆغزە بۆ ئەنجامی زانراولەسەرجۆغزەسەرەتاییکان کە پەیوەندیدارن بەداتاشراوەکانی $(\varphi، \theta)$ دەکەین کەبەبوونیان ئەم گشتاندنانەئەام دەدرێن.

## بعض تأثيرات اشتقاقات$(\varphi، \theta)$على الحلقات الاولية مركزيا

**عادل قادرجبار, قسم الرياضيات- كلية العلوم- جامعة السليمانية-اقليم كوردستان-العراق.**
**عبدالرحمن حميد مجيد, قسم الرياضيات – كلية العلوم – جامعة بغداد \ العراق.**

### الخلاصة

فى هذا البحث تمت دراسة بعض تاثيرات اشتقاقات $(\varphi، \theta)$على الحلقات الاولية مركزيا ونحاول تعميم بعض النتائج المعروفة على الحلقات الاولية والتى تتعلق باشتقاقات $(\varphi، \theta)$الى الحلقات الاولية مركزيا وكذلك نحدد الشروط التى عند توفرها تصبح هذه التعميمات ممكنة.

***Received on 16\8\2007Aacepted 20\3\2008***

***وەرگیراوە لە 16|8|2007 و پەسةندكرا لة 20|3|2008***